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Stability of periodic waves in shallow water

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Waves of small but finite amplitude in shallow water can occur as periodic wave trains of permanent shape in two known forms, either as Stokes waves for the shorter wavelengths or as cnoidal waves for the longer wavelengths. Calculations are made here of the periodic wave trains of permanent shape which span uniformly the range of increasing wavelength from Stokes waves to cnoidal waves and beyond. The present investigation is concerned with the stability of such permanent waves to periodic disturbances of greater or equal wavelength travelling in the same direction. The waves are found to be stable to infinitesimal and to small but finite disturbances of wavelength greater than the fundamental, the margin of stability decreasing either as the fundamental wave becomes more nonlinear (i.e. contains more harmonics), or as the wavelength of the periodic disturbance becomes large compared with the fundamental wavelength. The decreasing margin of stability is associated with an increasing loss of spatial periodicity of the wave train, to the extent that small but finite disturbances can cause a form of interaction between consecutive crests of the disturbed wave train. In such a case, a small but finite disturbance of wavelength n times the fundamental wavelength converts the wave train into n interacting wave trains. The amplitude of the disturbance subharmonic is then nearly periodic, the time scale being the time taken for repetitions of the pattern of interactions. When the disturbance is of the same wavelength as the permanent wave, the wave is found to be neutrally stable both to infinitesimal and to small but finite disturbances.

1. Introduction

Water waves of small but finite amplitude that propagate with constant shape are known as permanent waves. Periodic permanent waves include the Stokes wave (Stokes 1847) and the cnoidal wave (Korteweg & de Vries 1895). The Stokes wave has been shown by Benjamin (1967) to be linearly stable or unstable to periodic disturbances travelling in the same direction as the wave motion depending on whether the fundamental wavelength is greater or less respectively than 4·61 times the depth. The present investigation is concerned with the stability of periodic permanent waves to periodic disturbances travelling in the same direction in the range in which the Stokes wave is linearly stable. This is the range of periodic permanent waves whose wavelength is large compared with the depth.

The three length scales describing periodic permanent waves propagating in one direction along a uniform horizontal channel are the height $2a$ of the wave crest above the wave trough, the wavelength $2\pi l$ and the mean depth h. Two non-dimensional ratios may be formed from these lengths, namely $\epsilon = a/h$ and $\mu = h/l$. The approximation made in deriving the Stokes wave is that $\epsilon\mu \ll 1$ and that $\epsilon \ll \mu^2$ when $\mu < 1$. On the other hand, the approximation made in deriving cnoidal waves is that $\epsilon \ll 1$, with $\epsilon \sim \mu^2$. The incompatibility of the approximations prevents a uniform transition from Stokes waves to cnoidal waves for given $\epsilon \ll 1$ as μ decreases from 1 towards 0. The permanent waves which span uniformly this range are calculated below from the governing equations.

Since the displacement of the water surface is spatially periodic, it may be expanded in a Fourier series with time-dependent Fourier coefficients. The approximation to be made in § 2 in deriving equations for the Fourier coefficients is that $\epsilon \ll 1$ with $\mu < 1$. The permanent-wave solutions of these equations are found in § 3, where it is shown that these solutions span uniformly the range from Stokes waves through and beyond cnoidal waves for $\epsilon \ll 1$ as μ decreases from 1 towards 0. The stability of the permanent waves to periodic disturbances of wavelength greater than or equal to the fundamental wavelength is investigated in §§ 4–6.

The solutions are terminated at a time $t \sim \epsilon^{-2}$ at which the cumulative effect of the approximations may become significant. The solutions are valid, to the same approximation, for the spatial stability of a temporally periodic permanent wave generated by a wave maker oscillating at one end of an open uniform channel up to a distance $x \sim \epsilon^{-2}$ from the wave maker (Bona & Bryant 1973). In the latter context, the present investigation applies to the spatial stability of permanent waves to periodic disturbances of the wave maker of period greater than or equal to the fundamental period of the permanent wave.

The disturbance found by Benjamin (1967) to cause linear instability of Stokes waves consisted of a slow modulation of the fundamental wave train by a pair of wave modes with side-band frequencies and wavenumbers slightly different from the fundamental frequency and wavenumber. This property was also demonstrated by Whitham (1967) using a different method of analysis, and has been extended to two horizontal dimensions by Benney & Roskes (1969) and Hayes (1973). The instability results from resonant nonlinear interactions between the disturbing wave modes and the first two harmonics of the Stokes wave. A necessary condition for instability when the wave and the disturbance are travelling in the same direction is that $\mu > 1{\cdot}363$.

The nonlinear interactions between the harmonics of waves in shallow water are nearly resonant (Bryant 1973, § 2), the interactions becoming closer to resonance as μ decreases towards zero. Equivalently, the interactions between lower harmonics are closer to resonance than those between higher harmonics. Benjamin's analysis fails as $\mu \to 0$, the coefficients of equation (24) of Benjamin (1967), for example, being $O(\mu^{-3})$ as $\mu(\equiv \kappa) \to 0$, while they are required in his analysis to be $O(1)$. These coefficients become large compared with 1 as $\mu \to 0$ because the nonlinear interactions that generate them become closer to resonance as $\mu \to 0$. It is necessary to include more harmonics as $\mu \to 0$ in order to obtain the same accuracy in the analysis.

The response to periodic disturbances of permanent waves for which $\mu < 1$ may be anticipated from the principle of near-resonance. It is expected that the amplification of the disturbance increases either as μ decreases or as the wavelength of the disturbance increases because either effect causes the nonlinear interactions to become closer to resonance. Resonance itself cannot occur at the lowest order in ϵ for $\mu < 1{\cdot}363$, so instability in the strict sense does not occur. However, as $\mu \to 0$, or as the disturbance wavelength increases, it is expected that the amplification of the initial disturbance will become sufficiently large that permanent waves are modified substantially by small but finite periodic initial disturbances.

2. Equations of motion

Consider an irrotational flow bounded above by the free surface $y = \epsilon\eta(x,t)$ and below by the horizontal plane $y = -1$. The velocity potential $\phi(x,y,t)$ satisfies

$$\phi_{xx} + \mu^{-2}\phi_{yy} = 0, \tag{2.1a}$$

$$\phi_y = 0 \quad \text{on} \quad y = -1, \tag{2.1b}$$

$$\eta_t - \mu^{-2}\phi_y = \epsilon(\phi_t\phi_x)_x + O(\epsilon^2) \quad \text{on} \quad y = 0, \tag{2.1c}$$

$$\eta + \phi_t = \epsilon(-\tfrac{1}{2}\phi_x^2 - \tfrac{1}{2}\mu^{-2}\phi_y^2 + \phi_t\phi_{yt}) + O(\epsilon^2) \quad \text{on} \quad y = 0. \tag{2.1d}$$

The space co-ordinates x and y are measured in units of l and h respectively, and time t is measured in units of l/c_0, where $c_0 = (gh)^{\frac{1}{2}}$ is the linear long-wave velocity.

The periodic surface displacement and the velocity potential are expanded in the Fourier series

$$\eta(x,t) = \frac{1}{2}\sum_{k=1}^{\infty} B_k(t)\exp i(kx-\omega t) + \text{complex conjugate}, \tag{2.2}$$

$$\phi(x,y,t) = -\gamma t + \frac{1}{2}\sum_{k=1}^{\infty} C_k(t)\exp i(kx-\omega t)\cosh[\mu k(1+y)]/\cosh \mu k + \text{complex conjugate}, \tag{2.3}$$

where $\omega(k) = (k/\mu \tanh \mu k)^{\frac{1}{2}}$ is the angular frequency of infinitesimal waves of wavenumber k. It is noted from the linear theory that dB_k/dt, dC_k/dt and γ are $O(\epsilon)$. A more convenient Fourier series for $\eta(x,t)$ is

$$\eta(x,t) = \frac{1}{2}\sum_{k=1}^{\infty} A(k,t)\exp ik(x-ct) + \text{complex conjugate}, \tag{2.4}$$

where c is identified later with the phase velocity of permanent waves (in units of c_0).

When the Fourier series (2.2) and (2.3) are substituted into (2.1) and the amplitudes $C_k(t)$ eliminated, the equations for $B_k(t)$ stated by Bryant (1973) are obtained. The equations for the new Fourier amplitudes $A(k,t)$ are then

$$DA(k) - ik(c-\omega)A(k) = -i\epsilon\sum_{l=1}^{k-1}\tfrac{1}{2}S(k,l)A(l)A(k-l) - i\epsilon\sum_{l=1}^{\infty}R(k,l)A^*(l)A(k+l) + O(\epsilon^2), \quad k = 1,2,\ldots, \tag{2.5}$$

where $D = d/dt$,

$$R(k,l) = \frac{\{\omega(k+l)-\omega(l)\}\{l\omega(k+l)+(k+l)\,\omega(l)\}\,k+\omega(k)^2\,l(k+l)}{2\omega(l)\,\omega(k+l)\{\omega(k)+\omega(k+l)-\omega(l)\}} - \mu^2\frac{\omega(k)^2\{\omega(l)^2-\omega(l)\,\omega(k+l)+\omega(k+l)^2\}}{2\{\omega(k)+\omega(k+l)-\omega(l)\}}, \quad (2.6)$$

$$S(k,l) = R(k,-l) \quad \text{with} \quad \omega(-k) = -\omega(k). \quad (2.7)$$

There is no explicit restriction on μ in the derivation of (2.5). Note that the definitions of $R(k,l)$ and $S(k,l)$ have been written in such a form that these coefficients are defined for all real positive k and l, since this extension of the definitions from integral k and l is required when subharmonic disturbances of wavenumber less than 1 are introduced.

3. Permanent waves

The steady form of (2.5) with all harmonics in phase is

$$(kc-\omega)\,a_k = \epsilon\sum_{l=1}^{k-1}\tfrac{1}{2}S(k,l)\,a_l a_{k-l} + \epsilon\sum_{l=1}^{\infty}R(k,l)\,a_l a_{k+l} + O(\epsilon^2), \quad k = 1,2,\ldots, \quad (3.1)$$

where c and a_k $(k = 1,2,\ldots)$ are positive real variables. The permanent waves are the solutions of (3.1) subject to the constraint that the height of the crest above the trough is $2a$, namely

$$\sum_{k=1}^{\infty} a_{2k-1} = 1. \quad (3.2)$$

It is assumed, for the purpose of ordering the terms in (3.1), that for any given permanent wave the amplitudes approximate to a geometric sequence:

$$a_k \propto r^{k-1}. \quad (3.3)$$

Equations (3.1) are then solved by successive approximation, when it is found that the solutions are consistent with this ordering. The ratio r is dependent on ϵ/μ^2, with r increasing towards 1 as ϵ/μ^2 increases.

Stokes waves are the permanent waves for which $r \sim \epsilon$, when the only non-trivial equations obtained from (3.2) and (3.1) are

$$\left.\begin{aligned} a_1 &= 1+O(\epsilon^2),\\ (c-\omega(1))\,a_1 &= O(\epsilon^2),\\ (2c-\omega(2))\,a_2 &= \tfrac{1}{2}S(2,1)\,a_1^2+O(\epsilon^2).\end{aligned}\right\} \quad (3.4)$$

Their solution is

$$\left.\begin{aligned} a_1 &= 1+O(\epsilon^2),\\ a_2 &= \tfrac{1}{2}\epsilon S(2,1)/(2\omega(1)-\omega(2))+O(\epsilon^2)\\ &= \epsilon\mu(3-\tanh^2\mu)/4\tanh^3\mu+O(\epsilon^2),\\ c &= \omega(1)+O(\epsilon^2).\end{aligned}\right\} \quad (3.5)$$

This solution is the second approximation of Stokes (1847). The initial assumption that $r \sim \epsilon$ is seen from the solution to be equivalent to $\mu \sim 1$.

The third approximation of Stokes (1880) is found by keeping $r \sim \epsilon$ and by evaluating the terms $O(\epsilon^2)$, leaving a remainder $O(\epsilon^3)$. Laitone (1962) made this analysis in order to compare the higher approximations for Stokes waves with higher approximations for cnoidal waves. The aim of the present analysis is to calculate the periodic permanent waves which span uniformly the range from Stokes waves to cnoidal waves for fixed $\epsilon \ll 1$ as μ decreases from 1 towards 0. For this reason, smaller values of r are now taken and further harmonics are included in the equations in order to obtain solutions valid for smaller values of μ.

The next set of equations, for which $r^2 \sim \epsilon$, is

$$\left.\begin{aligned} a_1 + a_3 &= 1 + O(\epsilon^2), \\ (c - \omega(1))\, a_1 &= \epsilon R(1,1)\, a_1 a_2 + O(\epsilon^2), \\ (2c - \omega(2))\, a_2 &= \tfrac{1}{2}\epsilon S(2,1)\, a_1^2 + O(\epsilon^2), \\ (3c - \omega(3))\, a_3 &= \epsilon S(3,1)\, a_1 a_2 + O(\epsilon^2). \end{aligned}\right\} \tag{3.6}$$

Their solution is

$$c = \omega(1) + \tfrac{1}{2}\epsilon^2 R(1,1)\, S(2,1)/(2\omega(1) - \omega(2)) + O(\epsilon^2), \tag{3.7a}$$

$$a_1 = 1 - \tfrac{1}{2}\epsilon^2 S(2,1)\, S(3,1)/\{(2\omega(1) - \omega(2))\,(3\omega(1) - \omega(3))\} + O(\epsilon^2), \tag{3.7b}$$

$$a_2 = \tfrac{1}{2}\epsilon S(2,1)/(2\omega(1)) - \omega(2) + O(\epsilon^{\frac{3}{2}}), \tag{3.7c}$$

$$a_3 = 1 - a_1 + O(\epsilon^2). \tag{3.7d}$$

It may be shown from the dispersion relation that

$$\left.\begin{aligned} 2\omega(1) - \omega(2) &= \mu^2 + O(\mu^4), \\ 3\omega(1) - \omega(3) &= 4\mu^2 + O(\mu^4), \end{aligned}\right\} \tag{3.8}$$

so the initial assumption that $r^2 \sim \epsilon$ is seen from the solution to be equivalent to $\mu^4 \sim \epsilon$.

The set of equations containing the first harmonics is such that $r^{n-1} \sim \epsilon$. The set consists of $n+1$ equations for the $n+1$ variables $a_1, \ldots, a_n$ and c. Although analytical methods of solutions are possible for the smaller sets of equations, numerical methods must be used for the larger sets of equations. The generalized Newton method for systems of equations was tried and found successful, satisfactory convergence to the solution being achieved for up to 50 harmonics.

Some properties of the permanent-wave solutions for given $\epsilon = 0{\cdot}05$ as ϵ/μ^2 increases from 0·1 to 10·0 are sketched in figures 1–3. The value of ϵ was taken to be fixed at 0·05 and the independent variable was chosen to be ϵ/μ^2 in all calculations in the present investigation. The reason for this choice is that, in the Korteweg–de Vries model, long-wave systems are dependent only on the ratio ϵ/μ^2, and not on ϵ and μ independently. Although the present model is not as restrictive as the KdV model, it is closely approximated by it over much of the range of ϵ/μ^2, as is seen in figure 2.

Figure 1 shows the ratio a_2/a_1 of the first two harmonics. One of the main assumptions of Stokes's model is that the ratio a_2/a_1 should be small compared with 1. This assumption is seen to be true only for the smallest values of ϵ/μ^2 in the present range of values. The increase in the value of a_2/a_1 as e/μ^2 increases is a result of the interactions between the harmonics becoming closer to resonance as μ^2 decreases towards 0 at fixed small ϵ.

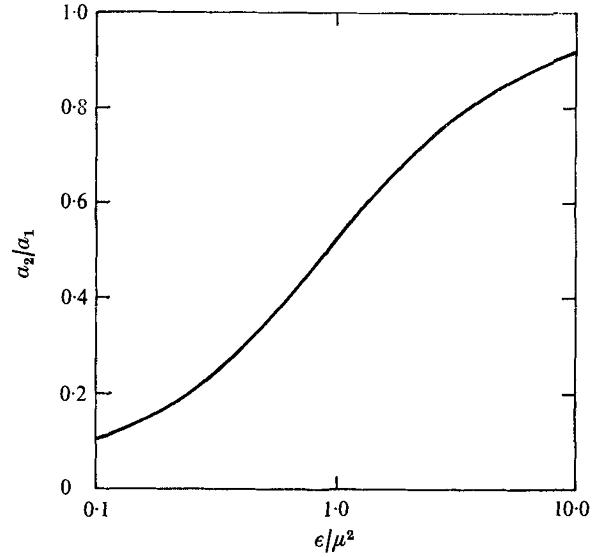

FIGURE 1. The ratio of the first two harmonics of periodic permanent waves on shallow water ($\epsilon = 0{\cdot}05$).

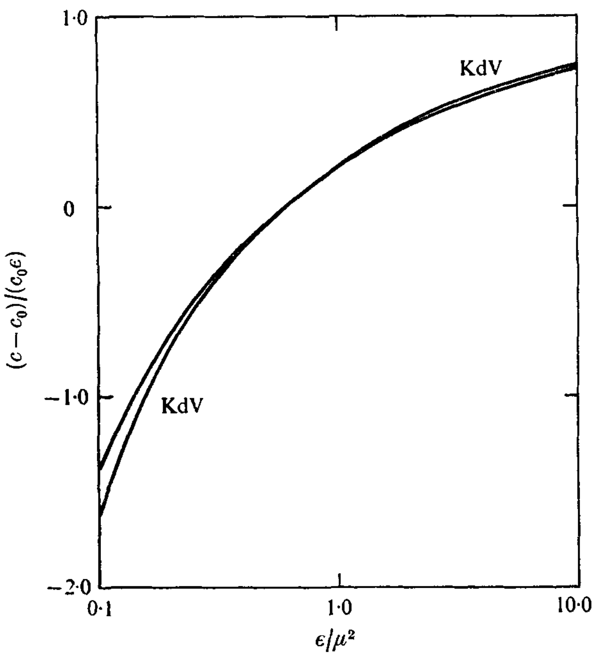

FIGURE 2. The phase velocity of periodic permanent waves compared with the phase velocity of the cnoidal waves of Korteweg & de Vries ($\epsilon = 0{\cdot}05$).

Figure 2 compares the phase velocity c of permanent waves with the phase velocity of the cnoidal waves of Korteweg & de Vries (Lamb 1945, §253). The latter is dependent only on the ratio ϵ/μ^2. The figure shows that divergence occurs at the two ends of the present range of values of ϵ/μ^2. The divergence for the small values of ϵ/μ^2 occurs because the basic assumption of the KdV model that $\mu^2 \ll 1$

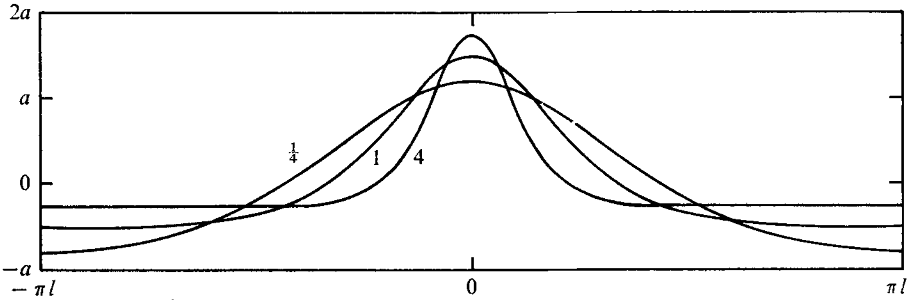

FIGURE 3. Profiles of one wavelength of a periodic permanent wave for each of three values of ϵ/μ^2 ($\epsilon = 0.05$).

is not valid there. The divergence for the large values of ϵ/μ^2 probably occurs because the KdV model underestimates the amplitudes of the higher harmonics, whose number and magnitude increase as ϵ/μ^2 increases.

Profiles of one wavelength of a periodic permanent wave for each of three values of ϵ/μ^2, namely $\frac{1}{4}$, 1 and 4, are sketched in figure 3. The nearly sinusoidal shape of the first corresponds close to a Stokes wave. As ϵ/μ^2 increases, the wave contains more harmonics, and its shape is such that each crest becomes more isolated from neighbouring crests. The increase in $\epsilon/\mu^2 \,(= al^2/h^3)$ corresponds usually to increasing the wavelength while the amplitude and depth are held constant, when the appearance of the wave train tends towards that of periodically spaced solitary waves.

4. Small periodic perturbations of permanent waves

The unperturbed permanent wave has the surface displacement

$$\eta(x,t) = \sum_{k=1}^{\infty} a_k \cos k\,(x-ct), \tag{4.1}$$

where a_k ($k = 1, 2, \ldots$) and c are the Fourier amplitudes and phase velocity. When the permanent wave is perturbed by a small periodic disturbance of wavenumber κ ($0 < \kappa < \frac{1}{2}$), the perturbed surface displacement is written as

$$\begin{aligned}\eta(x,t) = \tfrac{1}{2}\hat{A}(\kappa,t)\exp i\kappa(x-ct) + \frac{1}{2}\sum_{k=1}^{\infty} \{\hat{A}(k-\kappa,t)\exp[i(k-\kappa)(x-ct)] \\ + a_k \exp ik(x-ct) + \hat{A}(k+\kappa,t)\exp[i(k+\kappa)(x-ct)]\} + \text{complex conjugate}.\end{aligned} \tag{4.2}$$

The Fourier series does not include harmonics such as those with wavenumbers $k \pm 2\kappa$ generated by nonlinear interactions between two or more perturbation harmonics. In this sense, it is a linear expansion in the perturbation harmonics. Substitution of the perturbed harmonics in (2.5), followed by linearization in $\hat{A}$, yields

$$\begin{aligned}D\hat{A}(k-\kappa) - i\{(k-\kappa)c - \omega(k-\kappa)\}\hat{A}(k-\kappa) = -i\epsilon \sum_{l=1}^{k-1} S(k-\kappa, l)\, a_l \hat{A}(k-l-\kappa) \\ - i\epsilon \sum_{l=1}^{\infty} \{R(k-\kappa, l)\, a_l \hat{A}(k+l-\kappa) + R(k-\kappa, l-1+\kappa)\, a_{k+l-1}\hat{A}^*(l-1+\kappa)\} \\ + O(\epsilon^2), \quad k = 1, 2, \ldots,\end{aligned} \tag{4.3a}$$

$$D\hat{A}(k+\kappa)-i\{(k+\kappa)c-\omega(k+\kappa)\}\hat{A}(k+\kappa) = -i\epsilon\sum_{l=1}^{k} S(k+\kappa,l)\,a_l\hat{A}(k-l+\kappa)$$
$$-i\epsilon\sum_{l=1}^{\infty}\{R(k+\kappa,l)\,a_l\hat{A}(k+l+\kappa)+R(k+\kappa,l-\kappa)\,a_{k+l}\hat{A}^*(l-k)\}$$
$$+O(\epsilon^2),\quad k=0,1,2,\ldots. \quad (4.3b)$$

It is assumed, for the purpose of ordering the terms, that, for large k, $\{|\hat{A}(k\pm\kappa)|\}$ approximates to a geometric sequence with the same ratio as $\{a_k\}$ [equation (3.3)]. Equations (4.3) may then be solved by successive approximation.

The set of equations for $\hat{A}$ corresponding to equations (3.6) for a_k is

$$D\hat{A}(\kappa)-i\{\kappa c-\omega(\kappa)\}\hat{A}(\kappa) = -i\epsilon R(\kappa,1)\,a_1\hat{A}(1+\kappa)-i\epsilon R\,(\kappa,1-\kappa)\,a_1\hat{A}^*(1-\kappa), \quad (4.4a)$$

$$D\hat{A}(1-\kappa)-i\{(1-\kappa)c-\omega(1-\kappa)\}\hat{A}(1-\kappa) = -i\epsilon R(1-\kappa,\kappa)\,a_1\hat{A}^*(\kappa)$$
$$-i\epsilon R(1-\kappa,1)\,a_1\hat{A}(2-\kappa)-i\epsilon R(1-\kappa,1+\kappa)\,a_2\hat{A}^*(1+\kappa), \quad (4.4b)$$

$$D\hat{A}(1+\kappa)-i\{(1+\kappa)c-\omega(1+\kappa)\}\hat{A}(1+\kappa) = -i\epsilon S(1+\kappa,1)\,a_1\hat{A}(\kappa)$$
$$-i\epsilon R(1+\kappa,1)\,a_1\hat{A}(2+\kappa)-i\epsilon R(1+\kappa,1-\kappa)\,a_2\hat{A}^*(1-\kappa), \quad (4.4c)$$

$$D\hat{A}(2-\kappa)-i\{(2-\kappa)c-\omega(2-\kappa)\}\hat{A}(2-\kappa) = -i\epsilon S(2-\kappa,1)\,a_1\hat{A}(1-\kappa), \quad (4.4d)$$

$$D\hat{A}(2+\kappa)-i\{(2+\kappa)c-\omega(2+\kappa)\}\hat{A}(2+\kappa) = -i\epsilon S(2+\kappa,1)\,a_1\hat{A}(1+\kappa). \quad (4.4e)$$

This set of equations, augmented and approximated to $O(\epsilon^2)$, was used by Benjamin (1967, pp. 66–68) in his analysis of the stability of Stokes waves. Denoting by * the complex conjugate of an equation, as well as a variable, equations (4.4a, c, e, b^*, d^*) form a set of five first-order linear differential equations with constant coefficients for the five variables $\hat{A}(\kappa)$, $\hat{A}^*(1-\kappa)$, $\hat{A}(1+\kappa)$, $\hat{A}^*(2-\kappa)$ and $\hat{A}(2+\kappa)$.

The normal-mode solutions of the set of five equations are those solutions in which each of the five variables has the same time dependence $\exp(i\lambda\epsilon t)$. Substitution of this time dependence into (4.4a, c, e, b^*, d^*) allows the calculation of each λ as one of the five eigenvalues of the matrix of coefficients of these five equations. The set of equations is stable if all five values of λ are real, and unstable if any pair of values of λ are complex conjugates. The general solution of the set of equations may be calculated from the eigenvectors of the matrix.

For values of κ small compared with 1, the set of equations was found to be stable only for a finite range of values of ϵ/μ^2, the range being bounded below near the point predicted by Benjamin (1967) and bounded above near $\epsilon/\mu^2 = 0{\cdot}3$. Instability for the small values of ϵ/μ^2 was expected, because the set of equations (4.4) differs by only a few terms $O(\epsilon^2)$ from the set of equations used by Benjamin. Instability for $\epsilon/\mu^2 > 0{\cdot}3$ was suspect, since the set of five equations is not a satisfactory approximation near this value of ϵ/μ^2. When the set of equations extracted from (4.3) was enlarged by the inclusion of further harmonics, this upper bound for stability of the set was found to increase: the greater the number of harmonics included in the set of equations, the greater was their range of stability. Further investigation confirmed that the instability for $\epsilon/\mu^2 > 0{\cdot}3$ resulted from the near-resonance of the full set of equations (4.3) being approximated by instability in the truncated set of equations (4.4).

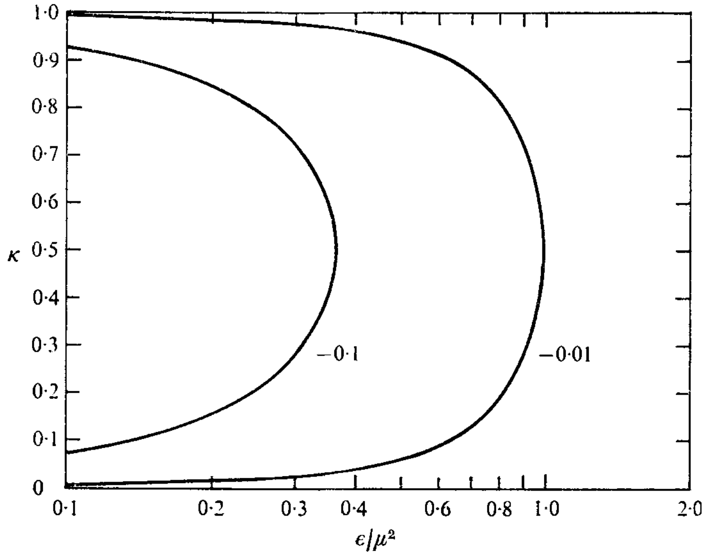

FIGURE 4. Stability diagram for periodic permanent waves ($\epsilon = 0{\cdot}05$). The curves are contours of the frequency of the lowest normal mode excited by a small periodic perturbation of wavenumber κ.

The set of equations for the first $2n+1$ perturbation harmonics $\hat{A}(\kappa), \hat{A}^*(1-\kappa)$, $\hat{A}(1+\kappa), \ldots, \hat{A}^*(n-\kappa)$, $\hat{A}(n+\kappa)$ consists of $2n+1$ linear first-order differential equations with constant coefficients obtained by truncation of ($4.3a^*$, b). The angular frequencies of the normal modes are the eigenvalues of the stability matrix formed from the coefficients of the $2n+1$ equations. The eigenvalues o the $(2n+1)\times(2n+1)$ matrix were found numerically by reducing the matrix to upper Hessenberg form and then using the QR algorithm (Wilkinson & Reinsch 1971, p. 359). For given ϵ and μ, the value of n was increased until the difference between the common eigenvalues of successive matrices was less than a small error (usually 10^{-3}). The general solution for this ϵ and μ was constructed from the eigenvectors of the final matrix, the eigenvectors being found numerically by a method of back substitution of the eigenvalues (Wilkinson & Reinsch 1971, p. 372).

The general solution for the perturbation harmonics is

$$\hat{A}^*(k-\kappa) = \sum_{j=1}^{2n+1} \alpha_j U_{2k,j} \exp(i\lambda_j \epsilon t), \quad k = 1, 2, \ldots, n, \tag{4.5a}$$

$$\hat{A}(k+\kappa) = \sum_{j=1}^{2n+1} \alpha_j U_{2k+1,j} \exp(i\lambda_j \epsilon t), \quad k = 0, 1, 2, \ldots, n, \tag{4.5b}$$

where the λ_j, $j = 1, \ldots, 2n+1$, are the $2n+1$ eigenvalues of the stability matrix in order of increasing magnitude, $\mathbf{U}$ is the $(2n+1)\times(2n+1)$ matrix whose columns are the $2n+1$ eigenvectors of the stability matrix, and the α_j, $j = 1, 2, \ldots, 2n+1$, are arbitrary constants determined by the initial conditions. Contours of λ_1 corresponding to values of $-0{\cdot}1$ and $-0{\cdot}01$ are sketched in figure 4 for $0{\cdot}1 < \epsilon/\mu^2 < 2{\cdot}0$ and $0 < \kappa < 1$. It is noted that $\lambda_1(1-\kappa) = \lambda_1(\kappa)$ for $0 < \kappa < \frac{1}{2}$, since the difference in the order of the lowest perturbation harmonics in the two calculations causes no significant differences in the eigenvalues. Values of $\lambda_1(\frac{1}{2})$ are calculated from equations (4.7) below. The magnitude of λ_1 is a measure of the margin of stability of the permanent wave for a given set of vaues of ϵ, μ and κ.

Examination of figure 4 shows that the margin of stability of a permanent wave decreases as κ decreases from $\frac{1}{2}$, that is, as the wavelength of a small periodic perturbation increases from twice the fundamental wavelength. The margin of stability of a permanent wave also decreases as ϵ/μ^2 increases. Both tendencies are expected, since the nonlinear interactions generating the perturbation harmonics become closer to resonance in both cases. With the present value of ϵ, the instability predicted by Benjamin (1967) occurs for $\epsilon/\mu^2 < 0{\cdot}027$, below the range covered by figure 4. It is intended to extend the present investigation by including terms $O(\epsilon^2)$ in order to continue figure 4 into the range in which instability occurs.

The Fourier series (4.2) and equations (4.3) contain redundant terms when $\kappa = \frac{1}{2}$. The Fourier series in this case is rewritten in the form

$$\eta(x,t) = \frac{1}{2}\sum_{k=1}^{\infty}\{\hat{A}(k-\tfrac{1}{2},t)\exp i(k-\tfrac{1}{2})(x-ct)+a_k\exp ik(x-ct)\} + \text{complex conjugate} \quad (4.6)$$

and the equations for $\hat{A}$ become

$$D\hat{A}(k-\tfrac{1}{2})-i\{(k-\tfrac{1}{2})c-\omega(k-\tfrac{1}{2})\}\hat{A}(k-\tfrac{1}{2}) = -i\epsilon\sum_{l=1}^{k-1}S(k-\tfrac{1}{2},l)\,a_l\hat{A}(k-l-\tfrac{1}{2})$$
$$-i\epsilon\sum_{l=1}^{\infty}\{R(k-\tfrac{1}{2},l)\,a_l\hat{A}(k+l-\tfrac{1}{2})+R(k-\tfrac{1}{2},l-\tfrac{1}{2})\,a_{k+l-1}\hat{A}^*(l-\tfrac{1}{2})\} + O(\epsilon^2), \quad k = 1,2,\ldots. \quad (4.7)$$

It is assumed for the purpose of ordering the terms that, for large k, $\{|\hat{A}(k-\frac{1}{2})|\}$ approximates to a geometric sequence with the same ratio as $\{a_k\}$. Equations (4.7) may then be solved by successive approximation.

The set of equations for the first n perturbation harmonics consists of $2n$ linear first-order differential equations with constant coefficients for the $2n$ variables $\hat{A}(\frac{1}{2})$, $\hat{A}^*(\frac{1}{2}), \ldots, \hat{A}(n-\frac{1}{2})$, $\hat{A}^*(n-\frac{1}{2})$. The n angular frequencies of the normal modes occur in pairs as equal and opposite real eigenvalues of the matrix of coefficients of the $2n$ equations formed by truncation from (4.7) and (4.7*). Although this investigation was continued to 40 perturbation harmonics with 80 equations, no evidence was found of linear instability of a permanent wave for $\kappa = \frac{1}{2}$, that is, for perturbations of twice the fundamental wavelength.

5. Small periodic perturbations of the same wavelength

The stability of a periodic permanent wave to small perturbations of the same wavelength is now examined. This is the case which has been analysed by Benjamin (1973, § 4.5), who showed that cnoidal waves are stable in shape to small but finite disturbances of the same wavelength. In terms of the stability diagram, figure 4, this case corresponds to $\kappa = 1$, or equivalently, to $\kappa = 0$. It is important because the solution clarifies the interpretation of the limit as $\kappa \to 0$, that is, as the wavelength of the perturbing wave becomes large compared with the wavelength of the permanent wave.

If the perturbation is chosen to be of the correct shape with the same phase and wavelength as the permanent wave, it can convert the permanent wave to another of slightly different shape and phase velocity. When the perturbed surface displacement is written as

$$\eta(x,t) = \frac{1}{2}\sum_{k=1}^{\infty}(a_k+\hat{A}(k))\exp ik(x-ct)+\text{complex conjugate}, \tag{5.1}$$

then for this particular perturbation,

$$(a_k+\hat{A}(k))\exp ik(x-ct) = (a_k+\delta a_k)\exp ik[x-(c+\delta c)t], \quad k = 1, 2, \ldots,$$

that is

$$\hat{A}(k) = (a_k+\delta a_k)\exp(-ik\delta ct)-a_k \tag{5.2a}$$

$$= \delta a_k - ika_k\delta ct + O(\delta^2). \tag{5.2b}$$

Such behaviour is usually known as neutral stability, since the growth of $\hat{A}(k)$ in time is due only to the change in phase velocity of the wave, the shape of the wave being stable in the usual sense.

Substitution of the perturbed harmonics into (2.5), followed by linearization in $\hat{A}$, yields

$$D\hat{A}(k)-i(kc-\omega(k))\hat{A}(k) = -i\epsilon\sum_{l=1}^{k-1}S(k,l)\,a_l\hat{A}(k-l)$$

$$-i\epsilon\sum_{l=1}^{\infty}R(k,l)\,(a_l\hat{A}(k+l)+a_{k+l}\hat{A}^*(l))+O(\epsilon^2), \quad k = 1, 2, \ldots. \tag{5.3}$$

It is assumed, for the purpose of ordering the terms, that, for large k, $\{|\hat{A}(k)|\}$ approximates to the same geometric sequence as $\{a_k\}$ [equation (3.3)]. The equations are then solved by successive approximation

The set of equations for $\hat{A}$ corresponding to equations (3.6) for a_k is

$$D\hat{A}(1)-i(c-\omega)1))\hat{A}(1) = -i\epsilon R(1,1)\,(a_1\hat{A}(2)+a_2\hat{A}^*(1)), \tag{5.4a}$$

$$D\hat{A}(2)-i(2c-\omega(2))\hat{A}(2) = -2i\epsilon S(2,1)\,a_1\hat{A}(1). \tag{5.4b}$$

(The equation for $\hat{A}(3)$ decouples from the set.) Equations (5.4*a*, *b*, *a**, *b**) form a set of four linear first-order differential equations with constant coefficients for the four variables $\hat{A}(1)$, $\hat{A}^*(1)$, $\hat{A}(2)$ and $\hat{A}^*(2)$. Elimination, followed by substitution for a_1 and a_2 from (3.7), leaves each of the four variables satisfying

$$D^2(D^2+\epsilon^2\lambda_2^2)\hat{A} = 0, \tag{5.5}$$

where 0 (twice) and $\pm\lambda_2$ are the eigenvalues of the matrix of coefficients of the four equations. The solution for $\hat{A}(1)$ is therefore

$$\hat{A}(1) = p_{11}+iq_{11}t+p_{12}\exp(i\epsilon\lambda_2 t)+q_{12}\exp(-i\epsilon\lambda_2 t),$$

where p_{11}, q_{11}, p_{12} and q_{12} are determined by the initial conditions. If the initial conditions are chosen to be $\hat{A}(1) = \hat{a}$ (complex) and $\hat{A}(2) = 0$, then p_{11} and q_{11} are both real and are consistent with (5.2*b*).

The set of equations for the first n perturbation harmonics consists of $2n$ linear first-order differential equations with constant coefficients for the $2n$ variables $\hat{A}(1), \hat{A}^*(1), \ldots, \hat{A}(n), \hat{A}^*(n)$ and is found by truncation from (5.3) and (5.3*). Their solution consists of a superposition of a neutrally stable contribution together with $2(n-1)$ normal modes having the time dependence $\exp(\pm i\lambda_j\epsilon t)$ with λ_j real, $j = 2, 3, \ldots, n$. The angular frequencies of the normal modes are the eigenvalues of the matrix of coefficients of the $2n$ equations, and include a doubles zero eigenvalue. The double zero eigenvalue was approximated in some instances by equal and opposite real numbers of small magnitude, and in other instances by complex-conjugate numbers of small magnitude, but apart from this curiosity of the approximations, all eigenvalues were found to be real. Although this investigation was continued to 40 perturbation harmonics with 80 equations, no evidence was found of linear instability of a permanent wave for $\kappa = 1$, that is, for perturbations of the same wavelength.

6. Examples

The three examples considered are the three profiles sketched in figure 3, namely the periodic permanent waves for which $\epsilon/\mu^2 = \frac{1}{4}$, 1 and 4. A range of values of κ was taken for each example, corresponding to different points on the stability diagram, figure 4. The solutions obtained by the linear stability analysi of §§4 and 5 were compared with direct numerical solutions of (2.5) without linearization in the perturbation amplitudes (Bryant 1973, §6). The results of these calculations are now summarized.

The case $\epsilon/\mu^2 = \frac{1}{4}$

The most stable perturbation is that with $\kappa = 0{\cdot}5$, of twice the wavelength of the permanent wave. The perturbation in this case consists of a lowest mode whose period is $2\pi/0{\cdot}178\epsilon$ and a much smaller second mode whose period is $2\pi/1{\cdot}208\epsilon$. When $\kappa = 0{\cdot}1$, the period of the lowest mode is $2\pi/0{\cdot}050\epsilon$, and when $\kappa = 0{\cdot}01$, it is $2\pi/0{\cdot}003\epsilon$. The amplification of the perturbation harmonics at side-band wavenumbers $1 \pm \kappa$ increases from about 2 at $\kappa = 0{\cdot}5$ to about 5 at $\kappa = 0{\cdot}1$, and to about 75 at $\kappa = 0{\cdot}01$. Although the linear solution in the latter case is strictly stable, the amplification is so large that for any reasonable initial perturbation the linear approximations fail. However, terms $O(\epsilon^2)$ are neglected in the analysis leading to this solution, so for consistency the exponentials of lowest degree should be replaced by their linear expansions. When this is done, the linear solution is recognized as describing nearly neutral stability. Each crest of the perturbed wave train moves with a nearly constant phase velocity which differs infinitesimally from the phase velocity of the crests on either side and from the phase velocity of the unperturbed wave train.

The case $\epsilon/\mu^2 = 1$

The perturbation when $\kappa = 0{\cdot}5$ consists of a lowest mode whose period is $2\pi/0{\cdot}0085\epsilon$ and a much smaller second mode whose period is $2\pi/0{\cdot}69\epsilon$. The amplification of the subharmonic of wavenumber $0{\cdot}5$ is about 25. The linear solution, with the exponentials of lowest degree replaced by their linear expansions, is in excellent agreement with the direct solution of (2.5) for $\hat{a} = 0{\cdot}01$, where $\hat{a}$ is the initial value of $\hat{A}(0{\cdot}5)$. Alternate crests of the perturbed wave train move with an infinitesimally larger velocity and intervening crests with an infinitesimally smaller velocity than the phase velocity of the undisturbed wave train, and a secondary wave of period $2\pi/0{\cdot}69\epsilon$ is superposed.

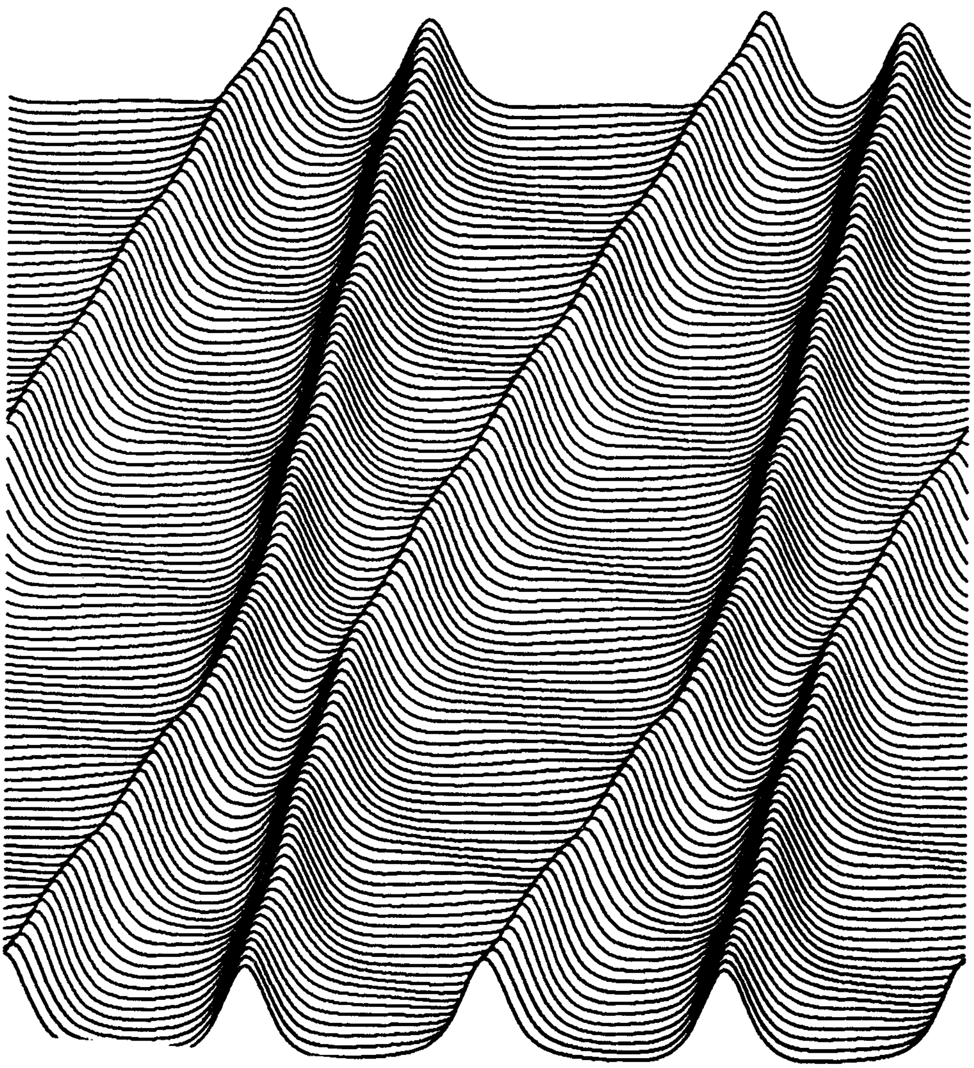

FIGURE 5. A periodic permanent wave disturbed by a small but finite perturbation of twice the wavelength ($\epsilon/\mu^2 = 1{\cdot}0$, $\kappa = 0{\cdot}5$, $\hat{a} = 0{\cdot}1$). The lines are graphs of $\eta(x, t)$ at values of ϵt $\frac{1}{8}\pi$ apart, $0 \leqslant \epsilon t \leqslant 20\pi$. Successive graphs are displaced vertically by a constant amount, with any part of a graph that lies below a previous graph being drawn on top of the previous graph.

This behaviour is magnified when $\hat{a} = 0{\cdot}1$ to the extent that the crests meet one another within the interval of direct integration of (2.5). The solution for $\eta(x, t)$ relative to a frame of reference moving with unit velocity is shown in figure 5. The figure can be viewed as a perspective drawing of $\eta(x, t)$ relative to the moving frame of reference, with x increasing over 4 wavelengths to the right and ϵt increasing by 20π away from the viewer. Apart from the form of the interactions, the appearance of the surface is similar to that of the examples illustrated previously (Bryant 1973). The two dominant wave trains meet more obliquely

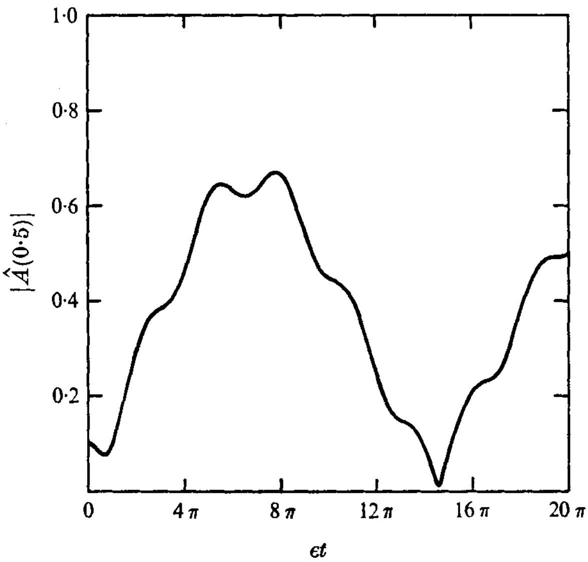

FIGURE 6. The amplitude of the subharmonic in the perturbed wave motion of figure 5.

than previously, with the result that consecutive crests do not cross over one another, but instead change roles at their point of closest approach. This property of waves changing roles was noticed by Zabusky (1967, p. 239) in the interaction of solitary waves, and by Madsen, Mei & Savage (1970) in the interaction of cnoidal waves. Alternate crests form separate primary and secondary wave trains, with phase discontinuities occurring between wave trains at their points of closest approach, although the paths of the crests remain continuous. A secondary wave of period about $2\pi/0{\cdot}7\epsilon$ is superposed.

This example is significant because it shows that the original permanent wave is stable in a nonlinear sense to small but finite perturbations. The original permanent wave is converted into two intersecting wave trains by small perturbations of twice the wavelength of the permanent wave. The subharmonic $\hat{A}(0{\cdot}5)$ is nearly periodic, the time scale being the time taken for repetitions of the pattern of intersections. The variation with time of the amplitude of the subharmonic is sketched in figure 6 for the same time interval $0 \leqslant \epsilon t \leqslant 20\pi$ as in figure 5. The amplitude of the subharmonic is a maximum at the closest approach of the wave crests and is a minimum when the wave trains return to a shape near to their unperturbed shape.

When the periodic perturbation is of the same wavelength as the permanent wave, $\kappa = 1$ or 0, and the linear solution for $\hat{A}(1)$ with initial conditions $\hat{A}(1) = \hat{a}$ (real) and $\hat{A}(k) = 0$ otherwise describes neutral stability in which the phase velocity of the perturbed wave train is $1 + 0{\cdot}211\epsilon + 0{\cdot}60\hat{a}\epsilon$ [equation (5.2*b*)], provided that $\hat{a}$ is sufficiently small. The direct solution for $\hat{A}(1)$ from (2.5) with $\hat{a} = 0{\cdot}01$ is sketched in figure 7 for $0 \leqslant \epsilon t \leqslant 5\pi$, and is in excellent agreement with the linear solution. When the initial perturbation is increased to $\hat{a} = 0{\cdot}1$, a direct solution of (2.5) shows that the original permanent wave is still neutrally stable, but now in a nonlinear sense, since the mean phase velocity is increased by a small

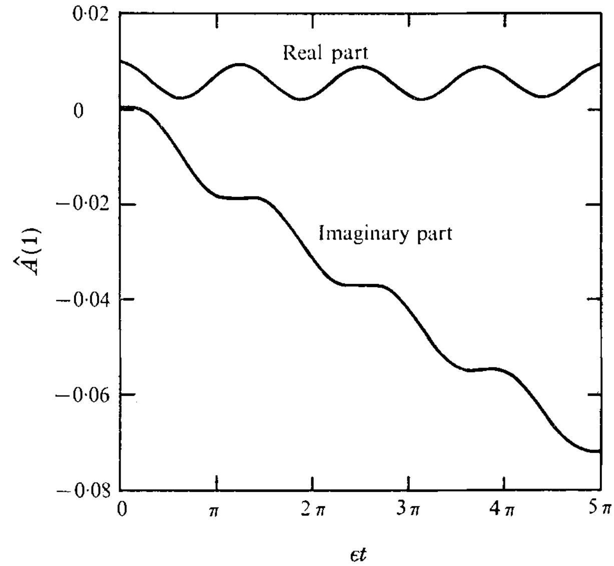

FIGURE 7. The perturbation of the first harmonic in a neutrally stable motion ($\epsilon/\mu^2 = 1$, $\kappa = 1$, $\hat{a} = 0{\cdot}01$).

but finite amount. The perturbation $\hat{A}(1)$ now takes the form of (5.2*a*), before the linearization to (5.2*b*). This result is in agreement with the analysis of Benjamin (1973, §4.5).

The case $\epsilon/\mu^2 = 4$

The periodic perturbations for which $\kappa = 0{\cdot}5$, $0{\cdot}1$ and $0{\cdot}01$ are all so close to neutral stability that it was not possible to separate numerically the lowest two eigenvalues from a double zero eigenvalue. For this reason, difficulties can be expected in generating or maintaining such a permanent wave. This was found to be the case in direct numerical solutions of (2.5), when the wave train tended to degenerate into a large primary and a much smaller secondary wave train within a few periods of starting the integration.

7. Discussion

The linearized perturbation analysis, together with the numerical integration of a large number of examples provide a comprehensive description of the stability of periodic permanent waves in shallow water to periodic disturbances travelling in the same direction. For permanent waves which are only weakly nonlinear ($\epsilon/\mu^2 < \frac{1}{2}$) perturbed by disturbances whose wavelength is only a small multiple of the fundamental wavelength ($0{\cdot}1 < \kappa < 0{\cdot}9$), the resulting wave motion is stable in the usual linear sense. For permanent waves which are more strongly nonlinear, or for disturbances whose wavelength is a larger multiple of the fundamental wavelength, the resulting wave motion is only marginally stable in the usual linear sense, the margin of stability being so small that for practical purposes it is zero. In such cases, the resulting wave motion is stable in a nonlinear sense, in that the disturbance is nearly periodic in time. This property has not been verified analytically, but was found to be true in all numerical examples evaluated. It was not appreciated in the first examples solved, when the large amplification of initial disturbances over the interval of integration led to the conclusion that the wave might be unstable. An extension of the interval of integration showed that even in these examples the disturbance is nearly periodic.

The analysis has been applied to the development in time of a wave train which is spatially periodic. Experimental investigations are usually concerned with the development in space of a wave train which is temporally periodic. Some deductions can be made about the experimental situation, since the two models are equivalent to within the approximations made in each case. Consider the experimental situation in which a periodic permanent wave is generated by a wave maker at one end of a uniform open channel. If the wave is only weakly nonlinear, it is linearly stable to slow modulation of the wave maker over moderate distances from the wave maker. For greater distances, the marginal stability of the wave train to even slower modulation of the wave maker makes itself apparent by an increasing erosion of the spatial periodicity of the wave train. If the wave is more strongly nonlinear, the marginal stability of the wave train to any slow modulation of the wave maker becomes apparent nearer to the wave maker. This in turn leads to consecutive crests overtaking one another and changing roles at their points of closest approach, within moderate distances of the wave maker. For large values of ϵ/μ^2, the margin of stability of the wave train is so small that experimental difficulties can be expected in generating a spatially periodic permanent wave.

REFERENCES

Benjamin, T. B. 1967 Instability of periodic wavetrains in nonlinear dispersive systems. *Proc. Roy. Soc.* A **299**, 59.

Benjamin, T. B. 1973 Lectures on nonlinear wave motion. *Fluid Mech. Res. Inst., University of Essex, Rep.* no. 44.

Benney, D. J. & Roskes, G. J. 1969 Wave instabilities. *Studies in Appl. Math.* **48**, 377.

Bona, J. L. & Bryant, P. J. 1973 A mathematical model for long waves generated by wavemakers in nonlinear dispersive systems. *Proc. Camb. Phil. Soc.* **73**, 391.

Bryant, P. J. 1973 Periodic waves in shallow water. *J. Fluid Mech.* **59**, 625.

Hayes, W. D. 1973 Group velocity and nonlinear dispersive wave propagation. *Proc. Roy. Soc.* A **332**, 199.

Korteweg, D. J. & de Vries, G. 1895 On the change of form of long waves advancing in a rectangular channel, and on a new type of long stationary waves. *Phil. Mag.* **39**, 422.

Laitone, E. V. 1962 Limiting conditions for cnoidal and Stokes waves. *J. Geophys. Res.* **67**, 1555.

Lamb, H. 1945 *Hydrodynamics*, Dover.

Madsen, O. S., Mei, C. C. & Savage, R. P. 1970 The evolution of time-periodic long waves of finite amplitude. *J. Fluid Mech.* **44**, 195.

Stokes, G. G. 1847 On the theory of oscillatory waves. *Trans. Camb. Phil. Soc.* **8**, 441. (See also *Papers*, vol. 1, p. 197.)

Stokes, G. G. 1880 *Mathematical and Physical Papers*, vol. 1, p. 314. Cambridge University Press.

Whitham, G. B. 1967 Non-linear dispersion of water waves. *J. Fluid Mech.* **27**, 399.

Wilkinson, J. H. & Reinsch, C. 1971 *Linear Algebra*. Springer.

Zabusky, N. J. 1967 *Nonlinear Partial Differential Equations* (ed. W. F. Ames). Academic.